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$W^{m,p}$ -Solution ($p \geq 2$) of Linear Degenerate Backward Stochastic Partial Differential Equations in the Whole Space[☆]

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Abstract

In this paper, we consider the backward Cauchy problem of linear degenerate stochastic partial differential equations. We obtain the existence and uniqueness results in Sobolev space $L^p(\Omega; C([0, T]; W^{m,p}))$ with both $m \geq 1$ and $p \geq 2$ being arbitrary, without imposing the symmetry condition for the coefficient σ of the gradient of the second unknown—which was introduced by Ma and Yong [17, Prob. Theor. Relat. Fields 113 (1999)] in the case of $p = 2$. To illustrate the application, we give a maximum principle for optimal control of degenerate stochastic partial differential equations.

Keywords: Degenerate backward stochastic partial differential equations, Cauchy problems, Sobolev spaces

1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space on which a d' -dimensional Wiener process $W = \{W_t; t \geq 0\}$ is defined such that $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by W , augmented by all the P -null sets in \mathcal{F} . We denote by \mathcal{P} the predictable σ -algebra associated with $\{\mathcal{F}_t\}_{t \geq 0}$. Let T be a fixed positive number.

We consider the backward Cauchy problem for the following linear stochastic partial differential equation (BSPDE in short)

$$\begin{aligned} du(t, x) = & - \sum_{i,j=1}^d \sum_{k=1}^{d'} \left\{ [a^{ij}(t, x)u_{x^j}(t, x) + \sigma^{ik}(t, x)q^k(t, x)]_{x^i} + b^i(t, x)u_{x^i}(t, x) \right. \\ & \left. + c(t, x)u(t, x) + \nu^k(t, x)q^k(t, x) + f(t, x) \right\} dt + \sum_{k=1}^{d'} q^k(t, x) dW_t^k, \quad (1.1) \\ (t, x) \in & [0, T) \times \mathbb{R}^d, \quad W_t \triangleq (W_t^1, \dots, W_t^{d'}), \quad t \geq 0 \end{aligned}$$

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and the terminal condition

$$u(T, x) = \varphi(x), \quad x \in \mathbb{R}^d. \quad (1.2)$$

They are a mathematically natural extension of backward stochastic differential equations (BSDEs) (see e.g. [8, 21]), whose solution consists of a pair of random fields (u, q) .

BSPDEs arise in many applications of probability theory and stochastic processes. For instance, in the optimal control of SDEs with partial information or more generally of stochastic parabolic PDEs, as adjoint equations of Duncan-Mortensen-Zakai filtering equations (see e.g. [4, 11, 19, 24, 25, 32]) to formulate the stochastic maximum principle for the optimal control, and in the formulation of the stochastic Feynman-Kac formula (see e.g. [16]) in mathematical finance. A class of fully nonlinear BSPDEs, the so-called backward stochastic Hamilton-Jacobi-Bellman equations, appear naturally in the dynamic programming theory of controlled non-Markovian processes (see [5, 10, 22]). For more aspects of BSPDEs, one can refer to [3, 6, 12, 18, 27, 28, 29].

In the case of the following *super-parabolicity*: the matrix

$$(A^{ij}) := \left(a^{ij} - \frac{1}{2} \sum_{k=1}^{d'} \sigma^{ik} \sigma^{jk} \right)_{d \times d} \geq \delta I_d \quad \text{for some } \delta > 0, \quad (1.3)$$

the theory of existence, uniqueness and regularity for solutions to BSPDE (1.1)-(1.2) is rather complete now. See among others [2, 4, 6, 7, 29, 31]. On the contrary, very few studies are devoted to BSPDEs (1.1)-(1.2) for the case of the *degenerate parabolicity*:

$$(A^{ij}) \geq 0. \quad (1.4)$$

Zhou [31] seemed to be the first to study degenerate BSPDEs, but under the assumption that the matrix $\sigma = (\sigma^{ik})_{d \times d'}$ vanishes. He ended up with commenting (see [31, page 340]) that “it remains a challenging open problem to solve the adjoint equations of degenerate SPDEs in which diffusion terms contain first-order differential operators” (in a dually equivalent term, the adjoint equations as BSPDEs contain the first order derivative of q , that is σ does not vanish). Seven years later, by introducing the following symmetry condition

$$\sum_{k=1}^{d'} \sigma^{ik} \frac{\partial \sigma^{jk}}{\partial x^l} = \sum_{k=1}^{d'} \sigma^{jk} \frac{\partial \sigma^{ik}}{\partial x^l}, \quad i, j, l = 1, \dots, d, \quad (1.5)$$

Ma and Yong [17] made a first step towards solution of the open challenging problem. In particular, assuming that the coefficient σ is invariant with the state variable x , which certainly satisfies the symmetry condition (1.5), Hu, Ma and Yong [12] further solved a semi-linear degenerate BSPDE for the case of $d = 1$. Note that counterexamples to the symmetry condition (1.5) are many, and here are three ones for the case of $d = d' = 2$ and $x := (x^1, x^2) \in \mathbb{R}^2$:

$$\sigma(x) = \begin{pmatrix} \sin(x^1 + x^2) & \cos(x^1 + x^2) \\ \cos(x^1 + x^2) & -\sin(x^1 + x^2) \end{pmatrix}, \quad (1.6)$$

$$\sigma(x) = \begin{pmatrix} \frac{1}{\sqrt{1+|x|^2}} & 1 \\ 0 & -\frac{1}{\sqrt{1+|x|^2}} \end{pmatrix}, \quad (1.7)$$

and

$$\sigma(x) = \begin{pmatrix} \sin(|x|) & \cos(|x|) \\ \cos(|x|) & -\sin(|x|) \end{pmatrix} \quad \text{with } |x| := \sqrt{(x^1)^2 + (x^2)^2}. \quad (1.8)$$

The preceding three works are basically restricted within the framework of the space $H^m = W^{m,2}$ with respect to the spatial variable $x \in \mathbb{R}^d$, with the only exception that Hu, Ma and Yong [12, Theorem 3.2, page 394] includes an L^p estimate for positive even number p . Using the method of stochastic flows, imposing neither any symmetry condition on σ nor any restriction on the dimension of \mathbb{R}^d , Tang [27] represented the solution of a system of semi-linear degenerate BSPDEs via the solutions of a family of FBSDEs and then established the existence and uniqueness of classical solutions to semi-linear systems of degenerate BSPDEs, but at a cost of assuming differentiability of higher orders in x on the coefficients. Some $W^{m,p}$ estimates are also given via the method of stochastic flows in Tang [26] for $m = 1, 2$ and $p \geq 2$. The last two works strongly suggest that the symmetry condition (1.5) be superfluous.

In this paper, we obtain the existence, uniqueness and regularity of generalized (or weak) solutions of linear degenerate BSPDEs under a quite general setting (see Theorem 2.1), in particular without imposing the symmetry condition (1.5). Besides, we give a $W^{m,p}$ -estimate with $m \geq 1$ for any $p \geq 2$. Our approach inherits the spirit of Krylov and Rozowskii [15] in the solution of degenerate stochastic PDEs, and invokes the real analysis lemma of Oleinik [20] in the study of linear degenerate parabolic PDEs so as to derive a priori estimates for degenerate BSPDEs. However, we have to develop some innovations to their calculus. The fundamental estimate of Krylov and Rozowskii [15, Lemma 2.1, page 339] is based on computing the quantity

$$\sum_{|\alpha| \leq m} G(|D^\alpha u|^2)$$

for the convex function $G(s) := s^p, s \geq 0$. Repetition of their calculus to our BSPDEs (3.2) finds the following new and trouble term:

$$\sum_{|\alpha| \leq m, \beta \leq \alpha} D^\alpha u D^\beta r G'(|D^\alpha u|^2),$$

which involves both unknowns u and r of our BSPDEs (3.2). Using Cauchy-Schwarz inequality, we only get the following

$$2D^\alpha u D^\beta r G'(|D^\alpha u|^2) \leq G'(|D^\alpha u|^2) (\epsilon |D^\beta r|^2 + \epsilon^{-1} |D^\alpha u|^2), \quad \beta < \alpha,$$

and it is difficult to see unless $p = 1$ that it is dominated by the following

$$\sum_{|\alpha| \leq m} (|D^\alpha u|^2 + |D^\alpha r|^2) G'(|D^\alpha u|^2)$$

due to the fact that $G'(|D^\alpha u|^2)$ varies with α for $p > 1$. To overcome this difficulty, we instead compute the following

$$G\left(\sum_{|\alpha| \leq m} |D^\alpha u|^2\right), \quad (1.9)$$

which leads to the following term

$$G' \left(\sum_{|\alpha| \leq m} |D^\alpha u|^2 \right) \sum_{|\alpha| \leq m, \beta \leq \alpha} D^\alpha u D^\beta r.$$

Since the derivative $G' \left(\sum_{|\alpha| \leq m} |D^\alpha u|^2 \right)$ is invariant with α for any $p \geq 1$, the sum is dominated by

$$G' \left(\sum_{|\alpha| \leq m} |D^\alpha u|^2 \right) \sum_{|\alpha| \leq m} (|D^\alpha u|^2 + |D^\alpha r|^2). \quad (1.10)$$

In this way, we introduce a series of innovational computations and develop some new techniques. For example, a localization method and some calculation techniques in application of Itô formula (see Remark 3.1 (i)) are given to deal with the difficult and subtle degenerate parabolicity condition.

As we know, BSPDEs arising from many applications are degenerately parabolic rather than being super-parabolic (see, e.g. [22, 16, 24, 25, 10]). Optimal control of SPDEs is solved in Section 4, so as to illustrate the application of our new result on degenerate parabolic BSPDE.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and state our main result (Theorem 2.1). In Section 3, we prove the main theorem, after some preparations starting from the super-parabolic BSPDE with smooth coefficients. In Section 4, a maximum principle for optimal control of degenerate SPDEs is formulated as an application of degenerate parabolic BSPDEs. In Sections 5 and 6, we give the proofs for two important lemmas which are used in the proof of Theorem 2.1.

2. Preliminaries and the main result

Let d and d' be two positive integers, and \mathbb{R}^d be d -dimensional Euclidean space whose squared norm is denoted by $\|\cdot\|$. Consider the following linear BSPDE

$$du = -(a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu + \sigma^{ik} q_{x^i}^k + \nu^k q^k + f) dt + q^k dW_t^k \quad (2.1)$$

with the terminal condition

$$u(T) = \varphi. \quad (2.2)$$

Here, the coefficients $a^{ij}(t, x) = a^{ji}(t, x)$, $b^i(t, x)$, $\sigma^{ik}(t, x)$, $\nu^k(t, x)$ for $i, j = 1, \dots, d$ and $k = 1, \dots, d'$, and $c(t, x)$, $f(t, x)$ and $\varphi(x)$ (as a rule, the argument ω is omitted), $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$, are given and measurable. The summation convention is in force for repeated indices throughout this paper.

To define the solution of BSPDE (2.1)-(2.2), we need more notations.

• Denote $u_x = \nabla u = (u_{x^1}, \dots, u_{x^d})$ for $x \in \mathbb{R}^d$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, denote

$$D^\alpha = \left(\frac{\partial}{\partial x^1} \right)^{\alpha_1} \left(\frac{\partial}{\partial x^2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x^d} \right)^{\alpha_d}$$

and $|\alpha| = \alpha_1 + \dots + \alpha_d$.

- Denote by C_0^∞ and C^n the set of infinitely differentiable real functions of compact support defined on \mathbb{R}^d and the set of n times continuously differentiable functions on \mathbb{R}^d such that

$$\|u\|_{C^n} := \sum_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} |D^\alpha u(x)| < \infty.$$

- For $p > 1$ and an integer $m \geq 0$, denote by $W^{m,p} = W^{m,p}(\mathbb{R}^d; \mathbb{R})$ the Sobolev space of real functions on \mathbb{R}^d with a finite norm

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \leq m} \int_{\mathbb{R}^d} |D^\alpha u|^p dx \right)^{\frac{1}{p}},$$

where α is a multi-index. In particular, $W^{0,p} = L^p$. It is well known that $W^{m,2}$ is a Hilbert space and denote its inner product by $\langle \cdot, \cdot \rangle_m$.

- For $p > 1$ and integer $m \geq 0$, denote by $W^{m,p}(d') = W^{m,p}(\mathbb{R}^d; \mathbb{R}^{d'})$ the Sobolev space of d' -dimensional vector-valued functions on \mathbb{R}^d , equipped with the norm $\|v\|_{m,p} = (\sum_{k=1}^{d'} \|v^k\|_{m,p}^p)^{1/p}$.

- Denote by $L^p_{\mathcal{D}} W^{m,p}$ (resp. $L^p_{\mathcal{D}} W^{m,p}(d')$) the space of all predictable process $u : \Omega \times [0, T] \rightarrow W^{m,p}$ (resp. $u : \Omega \times [0, T] \rightarrow W^{m,p}(d')$) such that $u(\omega, t) \in W^{m,p}$ (resp. $u(\omega, t) \in W^{m,p}(d')$) for a.e. (ω, t) and

$$\mathbb{E} \int_0^T \|u(t)\|_{m,p}^p dt < \infty.$$

- Denote by $L^p_{\mathcal{D}} C W^{m,p}$ (resp. $L^p_{\mathcal{D}} C_w W^{m,p}$) the space of all predictable process $u : \Omega \times [0, T] \rightarrow W^{m,p}$ strongly (resp. weakly) continuous w.r.t. t on $[0, T]$ for a.s. ω , such that

$$\mathbb{E} \sup_{t \leq T} \|u(t)\|_{m,p}^p < \infty.$$

Given an integer $m \geq 1$, we shall use the following assumptions.

(**A_m**) The given functions $a = (a^{ij})$, $b = (b^i)$, c , $\sigma = (\sigma^{ik})$ and $\nu = (\nu^k)$ are $\mathcal{P} \times B(\mathbb{R}^d)$ -measurable with values in \mathbb{S}^d (the set of real symmetric $d \times d$ matrices), \mathbb{R}^d , \mathbb{R} , $\mathbb{R}^{d \times d'}$ and $\mathbb{R}^{d'}$, respectively. The coefficients b^i, c, ν^k and their derivatives w.r.t. x up to the order m , as well as a^{ij} and σ^{ik} up to the order $\max\{2, m\}$, are uniformly bounded by the positive constant K_m .

(**P**) (*parabolicity*) For each $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$[2a^{ij}(t, x) - \sigma^{ik}\sigma^{jk}(t, x)]\xi^i\xi^j \geq 0, \quad \forall \xi \in \mathbb{R}^d.$$

(**F_m**) The function $f \in L^2_{\mathcal{D}} W^{m,2}$, the function φ is $\mathcal{F}_T \times B(\mathbb{R}^d)$ -measurable with $\varphi(\omega, \cdot) \in W^{m,2}$ for each ω and φ is \mathcal{F}_T -measurable as a function on Ω with values in $W^{m,2}$.

Definition 2.1. A pair of random fields $(u, q) \in L^2_{\mathcal{D}}W^{1,2} \times L^2_{\mathcal{D}}W^{0,2}(d')$ is called a generalized (or weak) solution of BSPDE (2.1)-(2.2) if for each $\eta \in C_0^\infty$ and a.e. t ,

$$\begin{aligned} \langle u(t), \eta \rangle_0 = & \langle \varphi, \eta \rangle_0 + \int_t^T \left[-\langle a^{ij}(s)u_{x^i}(s) + \sigma^{jk}(s)q^k(s), \eta_{x^j} \rangle_0 \right. \\ & + \langle (b^i(s) - a_{x^j}^{ij}(s))u_{x^i}(s) + c(s)u(s) + (\nu^k(s) - \sigma_{x^j}^{jk}(s))q^k(s) \\ & \left. + f(s), \eta \rangle_0 \right] ds - \int_t^T \langle q^k(s), \eta \rangle_0 dW_s^k \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (2.3)$$

Remark 2.1. If (u, q) is a generalized solution of BSPDE (2.1)-(2.2) and $(u, q) \in L^2_{\mathcal{D}}W^{n,2} \times L^2_{\mathcal{D}}W^{n-1,2}$, then u has a modification $\tilde{u} : \Omega \times [0, T] \rightarrow W^{n-1,2}$ which is continuous in t for all ω . Actually, this assertion is a backward version of Theorem 1.3.2 in [14] and follows immediately from Lemma 3.1 in [6]. For this, one needs take $H = W^{n-1,2}$, $V = W^{n,2}$ and $V^* = W^{n-2,2}$ in the referenced lemma and replace η by $(1 - \Delta)^{n-1}\eta$ in (2.3) with $\langle \cdot, \cdot \rangle_0$ accordingly changed into $\langle \cdot, \cdot \rangle_{n-1}$, due to the fact that $\langle v, (1 - \Delta)^{n-1}\eta \rangle_0 = \langle v, \eta \rangle_{n-1}$ for $v \in W^{n-1,2}$.

In the rest of this paper, we always take the continuous version of u .

Now we state our main theorem.

Theorem 2.1. *Let conditions (\mathbf{A}_m) , (\mathbf{P}) , and (\mathbf{F}_m) be satisfied for given $m \geq 1$. Then BSPDE (2.1)-(2.2) has a unique generalized solution (u, q) such that*

$$u \in L^2_{\mathcal{D}}C_w W^{m,2} \quad \text{and} \quad q + \nabla u \sigma \in L^2_{\mathcal{D}}W^{m,2}(d'),$$

and for any integer $m_1 \in [0, m]$, we have the estimates

$$\begin{aligned} \mathbb{E} \sup_{t \leq T} \|u(t)\|_{m_1,2}^2 + \mathbb{E} \int_0^T \|(q + \nabla u \sigma)(t)\|_{m_1,2}^2 dt \\ \leq C \mathbb{E} \left(\|\varphi\|_{m_1,2}^2 + \int_0^T \|f(t)\|_{m_1,2}^2 dt \right). \end{aligned} \quad (2.4)$$

In addition, if $f \in L^p_{\mathcal{D}}W^{m,p}$ and $\varphi \in L^p(\Omega, \mathcal{F}_T, W^{m,p})$ for $p \geq 2$, then $u \in L^p_{\mathcal{D}}C_w W^{m,p}$, and for any integer $m_1 \in [0, m]$,

$$\mathbb{E} \sup_{t \leq T} \|u(t)\|_{m_1,p}^p \leq C e^{Cp} \mathbb{E} \left(\|\varphi\|_{m_1,p}^p + \int_0^T \|f(t)\|_{m_1,p}^p dt \right). \quad (2.5)$$

Here and in the rest of this paper, C is a generic constant which depends only on d, d', K_m, m and T .

Remark 2.2. Theorem 2.1 asserts the existence and uniqueness of the solution of a degenerate BSPDE under quite general conditions, and improves the relevant results of Zhou [31], Ma and Yong [17] and Tang [27]. It is a rather satisfactory solution of the open problem posed by Zhou [31].

The proof of Theorem 2.1 is deferred to the next section. We now give some useful corollaries. By Sobolev's imbedding theorem (see e.g. [1]), we have

Corollary 2.2. *Under conditions of Theorem 2.1, (i) if $mp > d$, then the first component u is jointly continuous in (t, x) a.s.; (ii) if $2(m-2) > d$, then (u, q) is a classical solution of BSPDE (2.1)-(2.2) which also reads*

$$\begin{aligned} u(t, x) = & \varphi(x) + \int_t^T (a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu + \sigma^{ik}q_{x^i}^k + \nu^k q^k + f)(s, x) ds \\ & - \int_t^T q^k(s, x) dW_s^k \end{aligned} \quad (2.6)$$

holds for all $x \in \mathbb{R}^d$, $t \in [0, T]$ on a (t, x) -independent set of the full measure.

Proof. We only need to prove assertion (ii). In view of Theorem 2.1 and Sobolev's imbedding theorem, we can select appropriate versions such that $u(t, \cdot) \in C^2(\mathbb{R}^d)$ a.s., and $q(t, \cdot) \in C^1(\mathbb{R}^d)$, $f(t, \cdot), \varphi(\cdot) \in C^2(\mathbb{R}^d)$ for a.e. (ω, t) . Take a nonnegative function $\zeta \in C_0^\infty$ such that $\int_{\mathbb{R}^d} \zeta dx = 1$. For arbitrary $\varepsilon > 0$, we define the operator S_ε by

$$S_\varepsilon h(x) = \varepsilon^{-d} \zeta\left(\frac{x}{\varepsilon}\right) * h(x) \quad \text{for } x \in \mathbb{R}^d. \quad (2.7)$$

Since the convolution is representable as an inner product, from the definition of generalized solutions we have that

$$\begin{aligned} S_\varepsilon u(t, x) = & S_\varepsilon \varphi(x) + \int_t^T S_\varepsilon (a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu + \sigma^{ik}q_{x^i}^k + \nu^k + f)(s, x) ds \\ & - \int_t^T S_\varepsilon q^k(s, x) dW_s^k \end{aligned} \quad (2.8)$$

holds on a (t, x) -independent set of the full measure. Then in view of $q \in L_{\mathcal{F}}^2 W^{m-1,2}$, by the Burkholder-Davis-Gundy (BDG) inequality and Sobolev's theorem, we have

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \mathbb{E} \sup_{t \leq T} \left| \int_t^T [S_\varepsilon q^k(s, x) - q^k(s, x)] dW_s^k \right|^2 \\ & \leq C \lim_{\varepsilon \downarrow 0} \mathbb{E} \int_0^T \|S_\varepsilon q(s, \cdot) - q(s, \cdot)\|_{C^0}^2 ds \\ & \leq C \lim_{\varepsilon \downarrow 0} \mathbb{E} \int_0^T \|S_\varepsilon q(s) - q(s)\|_{m-2,2}^2 ds, \quad \forall x \in \mathbb{R}^d. \end{aligned}$$

In virtue of the properties of an averaging operator, the last limit vanishes. Thus, assertion (ii) follows immediately by taking $\varepsilon \downarrow 0$ in (2.8). \square

Remark 2.3. Corollary 2.2 relaxes the conditions of Tang [27]. Our proof of Corollary 2.2 (ii) is direct.

Moreover, Theorem 2.1 allows us to prove the boundedness of the first component of the solution and its derivatives, provided the coefficient f , the terminal data ξ and their derivatives are bounded.

Corollary 2.3. *Let conditions (\mathbf{A}_m) , (\mathbf{P}) , and (\mathbf{F}_m) with $m \geq 1$ be satisfied,*

$$f \in L^\infty_{\mathcal{D}} W^{m,\infty}, \quad \text{and} \quad \varphi \in L^\infty(\Omega, \mathcal{F}_T, W^{m,\infty}).$$

Then BSPDE (2.1)-(2.2) has a unique generalized solution (u, q) satisfying (2.4). Moreover, for arbitrary multi-index α s.t. $|\alpha| \leq m$, it holds that

$$|D^\alpha u(\omega, t, x)| \leq C$$

for almost all (ω, t, x) .

Proof. We only need to prove the last assertion. Set

$$N_p := (\mathbb{E} \|\varphi\|_{m,p}^p)^{1/p} + (\mathbb{E} \int_0^T \|f(t)\|_{m,p}^p dt)^{1/p}.$$

By interpolation, it holds that $f \in L^p_{\mathcal{D}} W^{m,p}$ and $\varphi \in L^p(\Omega, \mathcal{F}_T, W^{m,p})$ for each $p \geq 2$, and

$$N_p \leq N_2^{2/p} N_\infty^{1-2/p} \leq N_2 + N_\infty.$$

From Theorem 2.1 we have

$$\begin{aligned} \|D^\alpha u\|_{L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)} &\leq \sup_{p \geq 2} \left[\sup_{t \leq T} \left(\mathbb{E} \|u(t)\|_{m,p}^p \right)^{1/p} \right] \\ &\leq \sup_{p \geq 2} C p^{2/p} N_p \leq C(N_2 + N_\infty). \end{aligned}$$

The proof is complete. \square

3. Proof of Theorem 2.1

In this section, we prove Theorem 2.1. We begin with the following change of unknown variables in BSPDE (2.1):

$$r^k := q^k + \sigma^{ik} u_{x^i}.$$

Define

$$\alpha^{ij} := \frac{1}{2} \sigma^{ik} \sigma^{jk} \quad \text{and} \quad \tilde{b}^i := b^i - \sigma_{x^j}^{ik} \sigma^{jk} - \nu^k \sigma^{ik}. \quad (3.1)$$

Then BSPDE (2.1) can be rewritten into the following BSPDE:

$$\begin{aligned} du &= - \left[(a^{ij} - 2\alpha^{ij}) u_{x^i x^j} + \tilde{b}^i u_{x^i} + cu + \sigma^{ik} r_{x^i}^k + \nu^k r^k + f \right] dt \\ &\quad + (r^k - \sigma^{ik} u_{x^i}) dW_t^k, \end{aligned} \quad (3.2)$$

with the pair (u, r) being the unknown.

Take $G(\cdot) \in C^2[0, \infty)$ such that $G(s), G'(s) > 0$ and $G''(s) \geq 0$ for all $s \in (0, \infty)$. Applying Itô's formula formally to compute $G(\sum_{|\alpha| \leq m} |D^\alpha u|^2)$, we immediately have

$$\begin{aligned} dG\left(\sum_{|\alpha| \leq m} |D^\alpha u|^2\right) &= -\Theta(u, r, f; x, t, \omega) dt \\ &\quad + 2G'\left(\sum_{|\alpha| \leq m} |D^\alpha u|^2\right) \sum_{|\beta| \leq m} D^\beta u D^\beta (r^k - \sigma^{ik} u_{x^i}) dW_t^k, \end{aligned} \quad (3.3)$$

where

$$\begin{aligned}
& \Theta(u, r, f; x, t, \omega) \\
& := 2G' \left(\sum_{|\alpha| \leq m} |D^\alpha u|^2 \right) \sum_{|\beta| \leq m} D^\beta u D^\beta [(a^{ij} - 2\alpha^{ij})u_{x^i x^j} + \tilde{b}^i u_{x^i} + cu \\
& \quad + \sigma^{ik} r_{x^i}^k + \nu^k r^k + f] - G' \left(\sum_{|\alpha| \leq m} |D^\alpha u|^2 \right) \sum_{|\beta| \leq m} \|D^\beta (r - \sigma^i u_{x^i})\|^2 \\
& \quad - 2G'' \left(\sum_{|\alpha| \leq m} |D^\alpha u|^2 \right) \left\| \sum_{|\beta| \leq m} D^\beta u D^\beta (r - \sigma^i u_{x^i}) \right\|^2.
\end{aligned} \tag{3.4}$$

Note that the expression $\Theta(u, r, f; x, t, \omega)$ involves the leading coefficients a^{ij} and σ^{ik} in our degenerate parabolic equation. The following estimate turns out to be crucial in our arguments. In what follows, ε is a generic constant which can be chosen to be sufficiently small.

Lemma 3.1. *Let $m \geq 0, t \in [0, T]$ and $u \in W^{m+2,2}, r \in W^{m+1,2}(d')$. For $u \in W^{m,2}$ and $r \in W^{m,2}(d')$, define*

$$\Psi := \sum_{|\alpha| \leq m} |D^\alpha u|^2 \text{ and } \Upsilon := \sum_{|\alpha| \leq m} \|D^\alpha r\|^2. \tag{3.5}$$

Then under conditions (\mathbf{A}_m) and (\mathbf{P}) , we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \Theta(u, r, f; x, t, \omega) dx & \leq - (1 - \varepsilon) \int_{\mathbb{R}^d} G'(\Psi) \Upsilon dx + \frac{C}{\varepsilon} \int_{\mathbb{R}^d} [G(\Psi) + G'(\Psi) \Psi] dx \\
& \quad + \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G'(\Psi) |D^\beta f|^2 dx,
\end{aligned} \tag{3.6}$$

provided that every integral in the right-hand side is finite.

With the aid of Lemma 3.1, we are able to first establish, under the super-parabolic condition, the existence and uniqueness of the $W^{m,p}$ solution ($p \geq 2$) to BSPDEs with smooth coefficients. For this purpose, consider the following two assumptions.

(SP) (*super-parabolicity*) There is a constant $\delta > 0$ such that the matrix

$$2a - \sigma \sigma^* \geq \delta I_d.$$

(B) The coefficients $a^{ij}, b^i, c, \sigma^{ik}, \nu^k$ are infinitely differentiable in x for all (ω, t) , and their derivatives of orders up to n are dominated by a constant depending on n . Moreover, we assume that

$$f \in \bigcap_{m \in \mathbb{Z}_+} L^2_{\mathcal{F}} W^{m,2}, \quad \varphi \in \bigcap_{m \in \mathbb{Z}_+} L^2(\Omega, \mathcal{F}_T, W^{m,2}).$$

We always choose an appropriate modification of f such that for all t , $\mathbb{E} \|f(t)\|_{m,2}^2 < \infty$ for all $m \in \mathbb{Z}_+$.

We have the following lemma.

Lemma 3.2. Assume that $m \geq 0$, $p \geq 1$, $f \in L^2_{\mathcal{D}}W^{m,2p}$, and $\varphi \in L^{2p}(\Omega, \mathcal{F}_T, W^{m,2p})$. Then under conditions (\mathbf{A}_m) , (\mathbf{SP}) , and (\mathbf{B}) , BSPDE (2.1)-(2.2) has a unique generalized solution (u, q) such that

$$u \in L^{2p}_{\mathcal{D}}CW^{m,2p} \quad \text{and} \quad \Psi^{\frac{p-1}{2}}\Upsilon^{\frac{1}{2}} \in L^2_{\mathcal{D}}W^{0,2}, \quad (3.7)$$

with

$$\Psi := \sum_{|\alpha| \leq m} |D^\alpha u|^2 \quad \text{and} \quad \Upsilon := \sum_{|\alpha| \leq m} \|D^\alpha r\|^2 \quad (3.8)$$

for $r := q + u_x \sigma$. Moreover, we have

$$\begin{aligned} & \mathbb{E} \sup_{t \leq T} \int_{\mathbb{R}^d} \Psi^p dx + \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \Psi^{p-1} \Upsilon dx \\ & \leq C e^{Cp} \mathbb{E} \left(\|\varphi\|_{m,2p}^{2p} + \int_0^T \|f(t)\|_{m,2p}^{2p} dt \right), \end{aligned} \quad (3.9)$$

where the constant C is independent of δ and p .

Remark 3.1. (i) The proofs of Lemmas 3.1 and 3.2 are both quite technical and lengthy, and thus they are deferred to Sections 5 and 6, respectively. We would like to stress here that they play a key role in the proof of Theorem 2.1 and many calculation techniques are developed in their own proofs. For example, we apply Itô's formula in Lemma 3.1 to the expression

$$G \left(\sum_{|\alpha| \leq m} |D^\alpha u|^2 \right)$$

rather than to the expression $G(|D^\alpha u|^2)$ for each multi-index α which is conventionally used in the study of $W^{m,p}$ -theory of degenerate SPDEs (c.f. [15, 23]). It is difficult to derive a $W^{m,p}$ -estimate ($p \geq 2$) of solutions to BSPDEs by computing the expression $G(|D^\alpha u|^2)$ for each multi-index α , even under the super-parabolicity condition, we have to deal with the new terms $G' D^\alpha u D^\beta r$ with α and β being different multi-indices, and they turn out to be a trouble (see our more detailed exposition in the introduction). For another example, G is taken specifically for a localization method in the proof of Lemma 3.2 when applying Itô's formula to the solution of BSPDE. Though the stopping time arguments are conventionally used to get $W^{m,p}$ -estimate of solutions to SPDEs (c.f. [13, 23]), the localization method is conventionally used for BSPDEs and backward stochastic equations as well (see [30], where the localization method is used to get the $L^p(dx)$ solutions of backward doubly stochastic differential equations).

(ii) Treating the sum $q + \nabla u \sigma$ as a unity in our approach is a natural convenience of technical calculations, since it appears in the diffusion term of a BSDE (see e.g., [16, 27]).

(iii) By the equivalence of norms, (3.9) implies

$$\mathbb{E} \sup_{t \leq T} \|u(t)\|_{m,2p}^{2p} \leq C e^{Cp} \mathbb{E} \left(\|\varphi\|_{m,2p}^{2p} + \int_0^T \|f(t)\|_{m,2p}^{2p} dt \right)$$

with the same constant C as in (3.9).

Next we deal with the super-parabolic BSPDE with general coefficients.

Lemma 3.3. *Assume that $m \geq 0$, $p \geq 2$ and conditions (\mathbf{A}_m) , (\mathbf{SP}) and (\mathbf{F}_m) are satisfied. If $f \in L^p_{\mathcal{F}} W^{m,p}$ and $\varphi \in L^p(\Omega, \mathcal{F}_T, W^{m,p})$, then BSPDE (2.1)-(2.2) has a unique generalized solution (u, q) satisfying*

$$u \in L^2_{\mathcal{F}} W^{m+1,2} \cap L^2_{\mathcal{F}} C W^{m,2} \cap L^p_{\mathcal{F}} C_w W^{m,p} \text{ and } q \in L^2_{\mathcal{F}} W^{m,2}.$$

Moreover,

$$\begin{aligned} \mathbb{E} \sup_{t \leq T} \|u(t)\|_{m,2}^2 + \mathbb{E} \int_0^T \|(q + u_x \sigma)(t)\|_{m,2}^2 dt \\ \leq C \mathbb{E} \left(\|\varphi\|_{m,2}^2 + \int_0^T \|f(t)\|_{m,2}^2 dt \right) \end{aligned} \quad (3.10)$$

and

$$\mathbb{E} \sup_{t \leq T} \|u(t)\|_{m,p}^p \leq C e^{Cp} \mathbb{E} \left(\|\varphi\|_{m,p}^p + \int_0^T \|f(t)\|_{m,p}^p dt \right), \quad (3.11)$$

where the constant C is independent of δ and p .

Proof. In view of Du and Meng [6, Theorem 2.3] and Remark 2.1, BSPDE (2.1)-(2.2) has a unique generalized solution (u, q) satisfying

$$u \in L^2_{\mathcal{F}} W^{m+1,2} \cap L^2_{\mathcal{F}} C W^{m,2}, \quad q \in L^2_{\mathcal{F}} W^{m,2}.$$

Moreover,

$$\begin{aligned} \mathbb{E} \sup_{t \leq T} \|u(t)\|_{m,2}^2 + \mathbb{E} \int_0^T \left[\|u(t)\|_{m+1,2}^2 + \|q(t)\|_{m,2}^2 \right] dt \\ \leq C(\delta) \mathbb{E} \left(\|\varphi\|_{m,2}^2 + \int_0^T \|f(t)\|_{m-1,2}^2 dt \right). \end{aligned} \quad (3.12)$$

Applying the averaging operator S_ε defined by (2.7) to the functions $h = a^{ij}, b^i, c, \sigma^{ik}$ and ν^k , we set

$$h_n := S_{\frac{1}{n}} h, \quad f_n := S_{\frac{1}{n}} f, \quad \varphi_n := S_{\frac{1}{n}} \varphi.$$

In view of the properties of the averaging operator S_ε , we have the following assertions:

- (i) the functions $a_n^{ij}, b_n^i, c_n, \sigma_n^{ik}$ and ν_n^k satisfy the conditions of Lemma 3.2 for all n with the same constants K_m and δ ;
- (ii) for $h = a^{ij}, b^i, c, \sigma^{ik}, \nu^k$ and arbitrary multi-index α , we have

$$D^\alpha h_n(\omega, t, x) \rightarrow D^\alpha h(\omega, t, x), \quad \text{as } n \rightarrow \infty$$

uniformly w.r.t. (ω, t, x) ;

- (iii) for each n , both functions f_n and φ_n satisfy the conditions of Lemma 3.2, and for $p' = 2$ or p , we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\|\varphi_n - \varphi\|_{m,p'}^{p'} + \int_0^T \|f_n(t) - f(t)\|_{m,p'}^{p'} dt \right] = 0. \quad (3.13)$$

Consider the following BSPDE

$$\begin{cases} du_n = -(a_n^{ij}u_{n,xi}x^j + b_n^i u_{n,xi} + c_n u_n + \sigma_n^{ik} q_{n,xi}^k + \nu_n^k q_n^k + f_n) dt + q_n^k dW_t^k, \\ u_n(T) = \varphi_n. \end{cases} \quad (3.14)$$

The existence and uniqueness of the solution of (3.14) is indicated by Lemma 3.2. Moreover, in view of Lemma 3.2 (for $p = 1$), we have

$$\begin{aligned} \mathbb{E} \sup_{t \leq T} \|u_n(t)\|_{m,2}^2 + \mathbb{E} \int_0^T \|q_n(t) + \sigma_n^i u_{n,xi}(t)\|_{m,2}^2 dt \\ \leq C \mathbb{E} \left(\|\varphi_n\|_{m,2}^2 + \int_0^T \|f_n(t)\|_{m,2}^2 dt \right), \end{aligned} \quad (3.15)$$

where the constant C is independent of δ and n . To see this, set

$$\begin{aligned} \tilde{u}_n &:= u_n - u, \quad \tilde{q}_n := q_n - q, \quad \tilde{\varphi}_n := \varphi_n - \varphi, \\ \tilde{f}_n &:= (a_n^{ij} - a^{ij})u_{n,xi}x^j + (b_n^i - b^i)u_{n,xi} + (c_n - c)u \\ &\quad + (\sigma_n^{ik} - \sigma^{ik})q_{n,xi}^k + (\nu_n^k - \nu^k)q^k + f_n - f, \end{aligned}$$

then clearly $\tilde{f}_n \in L^2_{\mathcal{P}} W^{m-1,2}$. By the dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\|\tilde{\varphi}_n\|_{m,2}^2 + \int_0^T \|\tilde{f}_n(t)\|_{m-1,2}^2 dt \right] = 0.$$

Noting that $(\tilde{u}_n, \tilde{q}_n)$ is a generalized solution of BSPDE

$$\begin{cases} d\tilde{u}_n = -(a_n^{ij}\tilde{u}_{n,xi}x^j + b_n^i \tilde{u}_{n,xi} + c_n \tilde{u}_n + \sigma_n^{ik} \tilde{q}_{n,xi}^k + \nu_n^k \tilde{q}_n^k + \tilde{f}_n) dt + \tilde{q}_n^k dW_t^k \\ \tilde{u}_n(T) = \tilde{\varphi}_n \end{cases}$$

and applying a similar estimate as (3.12), we get

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \sup_{t \leq T} \|\tilde{u}_n(t)\|_{m,2}^2 + \int_0^T \left[\|\tilde{u}_n(t)\|_{m+1,2}^2 + \|\tilde{q}_n(t)\|_{m,2}^2 \right] dt \right\} = 0.$$

Passing $n \rightarrow \infty$ in (3.15), (3.10) follows.

Next we need to prove that $u \in L^p_{\mathcal{P}} C_w W^{m,p}$ and to deduce (3.11). We have known that $u_n, u \in L^2_{\mathcal{P}} C W^{m,2}$ and $\lim_{n \rightarrow \infty} \mathbb{E} \sup_{t \leq T} \|u_n(t) - u(t)\|_{m,2}^2 = 0$. Hence for arbitrary countable set $\{t_i\} \subset [0, T]$, $\eta_j \in C_0^\infty$ and multi-index α with $|\alpha| \leq m$, we have

$$\langle D^\alpha u(t_i), \eta_j \rangle_0 = \lim_{n' \rightarrow \infty} \langle D^\alpha u_{n'}(t_i), \eta_j \rangle_0 \leq \lim_{n' \rightarrow \infty} \|u_{n'}(t_i)\|_{m,p} \|\eta_j\|_{\frac{p}{p-1}} \quad (3.16)$$

on a full-measure set, where $\{n'\}$ is an appropriate subsequence of $\{n\}$.

Since $u \in L^2_{\mathcal{P}} C W^{m,2}$, it is easy to see that the left-hand side of (3.16) is continuous at t_i for each η_j on a full-measure set. Let t_i run through the rational points of $[0, T]$ and η_j run through a dense subset of the unit sphere of $L^{\frac{p}{p-1}}(\mathbb{R}^d)$. Then by (3.16) we obtain that for arbitrary $t \in [0, T]$ and multi-index α with $|\alpha| \leq m$,

$$\begin{aligned} \|D^\alpha u(t)\|_{0,p} &= \sup_j \langle D^\alpha u(t), \eta_j \rangle_0 \\ &\leq \sup_j \sup_i \langle D^\alpha u(t_i), \eta_j \rangle_0 \leq \lim_{n' \rightarrow \infty} \sup_{t \leq T} \|u_{n'}(t)\|_{m,p} \quad a.s. \end{aligned}$$

On the other hand, by Lemma 3.2 we know that

$$\mathbb{E} \sup_{t \leq T} \|u_{n'}(t)\|_{m,p}^p \leq C e^{Cp} \mathbb{E} \left(\|\varphi_{n'}\|_{m,p}^p + \int_0^T \|f_{n'}(t)\|_{m,p}^p dt \right). \quad (3.17)$$

Hence, in view of assertion (iii) in this proof and Fatou's lemma, we obtain (3.11). This estimate, along with the continuity of the left-hand side of (3.16) at t_i on a full-measure set and the separability of $L^{\frac{p}{p-1}}(\mathbb{R}^d)$, yields that $u(t) \in W^{m,p}$ for all t and $u(t)$ is weakly continuous in $W^{m,p}$ w.r.t. t on a full-measure set. The proof is complete. \square

Remark 3.2. It is worth noting that the $W^{m,p}$ estimate for $p \geq 2$ of Du, Qiu, and Tang [7] requires the super-parabolicity condition **(SP)**, and the generic constant C depends on δ of Condition **(SP)**. Our estimates (3.10) and (3.11) do not require such a dependence, but at a cost of losing the $W^{m+1,p}$ estimate.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Uniqueness. We need to prove that if $f = 0$, $\varphi = 0$ and (u, q) is a generalized solution of BSPDE (2.1)-(2.2), then $u = 0$ and $q = 0$ a.e. $(\omega, t) \in \Omega \times [0, T]$. Recalling Remark 2.1, we always choose the continuous modification of u and apply Itô's formula to $e^{Lt} \|u(t)\|_{0,2}^2$. Using Lemma 3.1 in the case that $G(s) = s^2$ and $m = 0$, we can easily get

$$e^{Lt} \|u(t)\|_{0,2}^2 \leq (C - L) \int_t^T e^{Ls} \|u(s)\|_{0,2}^2 ds + M_T - M_t,$$

where $M_t, t \in [0, T]$, is a continuous martingale with $M_0 = 0$ and the constant C comes from Lemma 3.1. By taking $L = C$, we have $0 \leq e^{Ct} \|u(t)\|_{0,2}^2 \leq M_T - M_t$, which implies that $u \equiv 0$ and then $q \equiv 0$. The uniqueness is proved.

Existence. Take $\varepsilon > 0$ and consider the following BSPDE

$$\begin{cases} du^\varepsilon = -(\varepsilon \Delta u^\varepsilon + a^{ij} u_{x^i x^j}^\varepsilon + b^i u_{x^i}^\varepsilon + c u^\varepsilon + \sigma^{ik} q_{x^i}^{\varepsilon,k} + \nu^k q^{\varepsilon,k} + f) dt + q^{\varepsilon,k} dW_t^k, \\ u^\varepsilon(T) = \varphi. \end{cases} \quad (3.18)$$

It is clear that BSPDE (3.18) satisfies the conditions of Lemma 3.3, thus (3.18) has a unique generalized solution $(u^\varepsilon, q^\varepsilon)$ satisfying all the assertions in Lemma 3.3. Set $r^\varepsilon = q^\varepsilon + \sigma^i u_{x^i}^\varepsilon$, then we have

$$\mathbb{E} \sup_{t \leq T} \|u^\varepsilon(t)\|_{m,2}^2 + \mathbb{E} \int_0^T \|r^\varepsilon(t)\|_{m,2}^2 dt \leq C \mathbb{E} \left(\|\varphi\|_{m,2}^2 + \int_0^T \|f(t)\|_{m,2}^2 dt \right),$$

where the constant C is independent of ε .

Therefore, we can find a sequence $\varepsilon_n \downarrow 0$ and

$$(u, r) \in L^2_{\mathcal{D}} W^{m,2} \times L^2_{\mathcal{D}} W^{m,2}(d')$$

such that $(u^{\varepsilon_n}, r^{\varepsilon_n})$ converges weakly in $L^2_{\mathcal{D}} W^{m,2} \times L^2_{\mathcal{D}} W^{m,2}(d')$ to (u, r) as $n \rightarrow \infty$. By the resonance theorem, we have

$$\mathbb{E} \int_0^T \|r(t)\|_{m,2}^2 dt \leq C \mathbb{E} \left(\|\varphi\|_{m,2}^2 + \int_0^T \|f(t)\|_{m,2}^2 dt \right). \quad (3.19)$$

If setting $q = r - u_x \sigma$, we know that q^{ε_n} converges weakly in $L^2_{\mathcal{D}} W^{m-1,2}(d')$ to q . Therefore, it is obvious that for any $\eta \in C_0^\infty$, all terms on both sides of (2.3) with (u, q) replaced by $(u^{\varepsilon_n}, q^{\varepsilon_n})$, converge weakly in $L^2_{\mathcal{D}}(\Omega \times [0, T])$ to the corresponding terms for (u, q) as $\varepsilon_n \downarrow 0$. Since the operators of integration and stochastic integration are continuous in $L^2_{\mathcal{D}}(\Omega \times [0, T])$, they are weakly continuous and thus (u, q) is a generalized solution of (2.1)-(2.2). Furthermore, noting Remark 2.1, we can always take $u \in L^2_{\mathcal{D}} C W^{m-1,2}$.

Since u^{ε_n} converges weakly in $L^2_{\mathcal{D}} W^{m,2}$ to u as $n \rightarrow \infty$, by Banach-Saks theorem we construct a sequence u^k from finite convex combinations of u^{ε_n} such that u^k converges strongly to u in $W^{m,2}$ for a.e. $t \in [0, T]$ a.s. Hence for some countably dense set $\{t_i\} \subset [0, T]$, any $\eta_j \in C_0^\infty$ and $|\alpha| \leq m$, we have

$$\begin{aligned} (-1)^{|\alpha|} \langle u(t_i), D^\alpha \eta_j \rangle_0 &= \langle D^\alpha u(t_i), \eta_j \rangle_0 = \lim_{k \rightarrow \infty} \langle D^\alpha u^k(t_i), \eta_j \rangle_0 \\ &\leq \lim_{k \rightarrow \infty} \|u^k(t_i)\|_{m,p} \|\eta_j\|_{\frac{p}{p-1}} \end{aligned} \quad (3.20)$$

on a full-measure set. Since $u \in L^2_{\mathcal{D}} C W^{m-1,2}$, we know that for any η_j , the first term of (3.20) is continuous at t_i on a full-measure set. Allow $\eta_j \in C_0^\infty$ to run through a dense subset of the unit sphere of $L^{\frac{p}{p-1}}(\mathbb{R}^d)$, by (3.20) we obtain that for $|\alpha| \leq m$,

$$\begin{aligned} \|D^\alpha u(t)\|_{0,p} &= \sup_j \langle D^\alpha u(t), \eta_j \rangle_0 \\ &\leq \sup_j \sup_i \langle D^\alpha u(t_i), \eta_j \rangle_0 \leq \lim_{k \rightarrow \infty} \sup_{t \leq T} \|u^k(t)\|_{m,p} \quad a.s. \end{aligned}$$

and for all t ,

$$\sup_{t \leq T} \|D^\alpha u(t)\|_{0,p} \leq \lim_{k \rightarrow \infty} \sup_{t \leq T} \|u^k(t)\|_{m,p} \quad a.s.$$

On the other hand, noting that u^{ε_n} satisfies (3.11), by Jensen's inequality we have

$$\mathbb{E} \sup_{t \leq T} \|u^k(t)\|_{m,p}^p \leq C e^{C_p} \mathbb{E} \left(\|\varphi\|_{m,p}^p + \int_0^T \|f(t)\|_{m,p}^p dt \right).$$

Therefore, (2.5) follows from Fatou's lemma. Similarly, we can establish the same estimate for $p = 2$ which, along with (3.19), yields (2.4). Using a similar argument as in the end of the proof of Lemma 3.3, we can deduce $u \in L^2_{\mathcal{D}} C_w W^{m,2} \cap L^p_{\mathcal{D}} C_w W^{m,p}$. The proof is complete. \square

4. Application: maximum principle for optimal control of degenerate SPDEs

One main application of BSPDEs is to formulate the maximum principle for optimal controls of SPDEs (c.f. [4, 24, 32]). Since Theorem 2.1 does not require the super-parabolic condition, we give an example to illustrate the application in stochastic control theory.

Let Γ be a non-empty Borel set in some Euclidean space. For $x \in \mathbb{R}^d$ and $\phi \in C_0^\infty$, we define two differential operators $\{\mathcal{L}(t, v) : t \in [0, T], v \in \Gamma\}$ and $\{\mathcal{M}^k(t, v) : t \in [0, T], v \in \Gamma, k = 1, 2, \dots, d'\}$ as follows:

$$\begin{aligned} \mathcal{L}(t, v) \phi(x) &:= [a^{ij}(t, x, v) \phi_{x^j}(x)]_{x^i} + b^i(t, x, v) \phi_{x^i}(x) + c(t, x, v) \phi(x), \\ \mathcal{M}^k(t, v) \phi(x) &:= \sigma^{ik}(t, x, v) \phi_{x^i}(x) + v^k(t, x, v) \phi(x), \end{aligned} \quad (4.1)$$

where $a^{ij}, b^i, c, \sigma^{ik}$ and v^k are given real valued functions defined in $[0, T] \times \mathbb{R}^d \times \Gamma$, $i, j = 1, 2, \dots, d$ and $k = 1, 2, \dots, d'$. In the mean time, we write down the adjoint operators of $\mathcal{L}(t, v)$ and $\mathcal{M}^k(t, v)$:

$$\begin{aligned}\mathcal{L}^*(t, v)\phi(x) &:= [a^{ij}(t, x, v)\phi_{x^j}(x)]_{x^i} - [b^i(t, x, v)\phi(x)]_{x^i} + c(t, x, v)\phi(x) \\ \mathcal{M}^{k*}(t, v)\phi(x) &:= -[\sigma^{ik}(t, x, v)\phi(x)]_{x^i} + v^k(t, x, v)\phi(x).\end{aligned}\tag{4.2}$$

We denote by \mathcal{V}_{ad} the totality of admissible controls which are Γ -valued, \mathcal{F} -adapted processes $\{V(t), 0 \leq t \leq T\}$.

Problem. Given $V(\cdot) \in \mathcal{V}_{ad}$, we consider the following controlled linear SPDE (with the state variable x being omitted):

$$\begin{cases} d\xi(t) &= [\mathcal{L}(t, V(t))\xi(t) + F(t, V(t))] dt \\ &\quad + [\mathcal{M}^k(t, V(t))\xi(t) + G^k(t, V(t))] dW_t^k, \quad t \in [0, T]; \\ \xi(0) &= \xi_0. \end{cases}\tag{4.3}$$

The optimal control problem is to find $V(\cdot) \in \mathcal{V}_{ad}$ which minimizes the cost functional below:

$$J(V) := \mathbb{E} \left\{ \int_0^T \langle f(t, V(t)), \xi^V(t) \rangle_0 dt + \langle \varphi, \xi^V(T) \rangle_0 \right\},\tag{4.4}$$

where $f : [0, T] \times \Gamma \rightarrow W^{1,2}$ and $\varphi \in W^{1,2}$ are given.

A process $\xi = \xi^V \in L^2_{\mathcal{D}} W^{1,2}$ is called a (generalized) solution of SPDE (4.3) for the control V if, for each $\eta \in C_0^\infty$ and a.e. $t \in [0, T]$,

$$\begin{aligned}\langle \xi(t), \eta \rangle_0 &= \langle \xi_0, \eta \rangle_0 + \int_0^t \langle \mathcal{L}(s, V(s))\xi(s) + F(s, V(s)), \eta \rangle_0 ds \\ &\quad + \int_0^t \langle \mathcal{M}^k(s, V(s))\xi(s) + G^k(s, V(s)), \eta \rangle_0 dW_s^k, \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

For this problem, we make the following hypotheses.

- The functions $a^{ij}, b^i, c, \sigma^{ik}, \nu^k : [0, T] \times \mathbb{R}^d \times \Gamma \rightarrow \mathbb{R}$ are measurable in (t, x, v) and continuous in v ; a^{ij}, σ^{ik} and their derivatives w.r.t. x up to second order, as well as b^i, c, ν^k and their first order derivatives, do not exceed a constant K_1 in absolute value.

- (*parabolicity*) For each $(t, x, v) \in [0, T] \times \mathbb{R}^d \times \Gamma$, the matrix

$$(2a^{ij} - \sigma^{ik}\sigma^{jk})_{d \times d} \geq 0.$$

- The function $f(t, v) \in W^{1,2}$ for each (t, v) , the function $\varphi \in W^{1,2}$ and

$$\|f(t, v)\|_{1,2} + \|\varphi\|_{1,2} \leq K_1.$$

- The functions $F, G^k : [0, T] \times \mathbb{R}^d \times \Gamma \rightarrow \mathbb{R}$ are measurable in (t, x, v) and continuous in v , furthermore, $F(t, v) \in W^{1,2}$ and $G^k(t, v) \in W^{2,2}$ for each (t, v) , and

$$|F(t, x, v)| + |G^k(t, x, v)| + \|F(t, \cdot, v)\|_{1,2} + \|G^k(t, \cdot, v)\|_{2,2} \leq K_1.$$

- $\xi_0 \in W^{1,2}$.

Under above conditions, we know immediately from Krylov and Rozowskii [15] that the state equation (4.3) has a unique solution $\xi^V \in L^2_{\mathcal{P}} C_w W^{1,2}$ for any given $V \in \mathcal{V}_{ad}$.

The adjoint equation of SPDE (4.3) reads

$$\begin{cases} du = -[\mathcal{L}^*(t, V)u + \mathcal{M}^{k*}(t, V)q^k + f(t, V)] dt + q^k dW_t^k, \\ u(T) = \varphi. \end{cases} \quad (4.5)$$

By Theorem 2.1, for any given $V \in \mathcal{V}_{ad}$, BSPDE (4.5) has a unique solution $(u, q) \in L^2_{\mathcal{P}} C_w W^{1,2} \times L^2_{\mathcal{P}} W^{0,2}(d')$.

Now we can give the necessary condition of an optimal control for the general SPDE system (4.3) with the cost functional (4.4).

Proposition 4.1. *Under above hypotheses, we assume that \tilde{V} is an optimal control along with the corresponding optimal state $\tilde{\xi}$. Then for a.e. $t \in [0, T]$, we have the maximum condition*

$$H(t, \tilde{\xi}(t), \tilde{V}(t), u(t), q(t)) = \max_{v \in \Gamma} H(t, \tilde{\xi}(t), v, u(t), q(t)), \quad \mathbb{P}\text{-a.s.},$$

where (u, q) is the solution of (4.5) with $V(t) = \tilde{V}(t)$ and the Hamiltonian function H is defined by

$$\begin{aligned} H(t, \phi, v, \zeta, \eta) &:= -\langle \mathcal{L}(t, v)\phi, \zeta \rangle_0 - \langle F(t, v), \zeta \rangle_0 \\ &\quad - \langle \mathcal{M}^k(t, v)\phi, \eta^k \rangle_0 - \langle G^k(t, v), \eta^k \rangle_0 - \langle f(t, v), \phi \rangle_0 \end{aligned}$$

for $(t, \phi, v, \zeta, \eta) \in [0, T] \times W^{1,2} \times \Gamma \times W^{1,2} \times L^2(d')$.

The procedure to the proof of Proposition (4.1) is very similar to Zhou [32, Theorem 5.1]. Although the adjoint equation BSPDE (4.5) only satisfies the degenerate parabolic condition rather than super-parabolic condition, Theorem 2.1 guarantees that degenerate parabolic condition still works.

It is well known that the optimal control problem of partially observed diffusions with general nonlinear cost functionals can be transformed into an optimal control with complete observation of Zakai's equations with linear cost functionals. Therefore the previous results also enable us to discuss the partially observed diffusions under degenerate parabolic condition.

5. Proof of Lemma 3.1

Define

$$I := \int_{\mathbb{R}^d} G'(\Psi) \Upsilon dx \quad \text{and} \quad J := \int_{\mathbb{R}^d} [G(\Psi) + G'(\Psi)\Psi] dx. \quad (5.1)$$

In what follows, the notations $G(\Psi)$, $G'(\Psi)$ and $G''(\Psi)$ will be occasionally simplified as G , G' and G'' in the following arguments, when no confusion occurs.

Now we estimate all terms in (3.4). First of all, we have

$$2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' D^\beta u D^\beta(cu) dx \leq C \int_{\mathbb{R}^d} G' \Psi dx \leq C J. \quad (5.2)$$

Then by Young's inequality, we have

$$\begin{aligned}
2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' D^\beta u D^\beta f \, dx &\leq \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} (|D^\beta u|^2 + |D^\beta f|^2) G' \, dx \\
&\leq J + \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' |D^\beta f|^2 \, dx.
\end{aligned} \tag{5.3}$$

By the integration by parts, it is not hard to get

$$\begin{aligned}
2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' D^\beta u D^\beta (\tilde{b}^i u_{x^i}) \, dx &\leq 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} \tilde{b}^i G' D^\beta u D^\beta u_{x^i} \, dx + C J \\
&\leq \int_{\mathbb{R}^d} \tilde{b}^i G_{x^i} \, dx + C J \leq - \int_{\mathbb{R}^d} \tilde{b}_{x^i}^i G \, dx + C J \leq C J.
\end{aligned} \tag{5.4}$$

Similarly as (5.4), it follows

$$-2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' (a^{ij} - 2\alpha^{ij})_{x^j} (D^\beta u) D^\beta u_{x^i} \, dx \leq C J.$$

Then we introduce a new notation $\sum_\beta := \sum_{\beta_1 + \beta_2 = \beta, |\beta_1| = 1}$ to deal with the second order derivatives of u and we have

$$\begin{aligned}
&2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' D^\beta u D^\beta [(a^{ij} - 2\alpha^{ij}) u_{x^i x^j}] \, dx \\
&\leq C J + 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' (a^{ij} - 2\alpha^{ij}) D^\beta u D^\beta u_{x^i x^j} \, dx \\
&\quad + 2 \sum_{|\beta| \leq m} \sum_{\beta} \int_{\mathbb{R}^d} G' D^\beta u D^{\beta_1} (a^{ij} - 2\alpha^{ij}) D^{\beta_2} u_{x^i x^j} \, dx \\
&\leq C J - 2 \int_{\mathbb{R}^d} G' (a^{ij} - 2\alpha^{ij}) D^\beta u_{x^i} D^\beta u_{x^j} \, dx \\
&\quad - 4 \int_{\mathbb{R}^d} G'' (a^{ij} - 2\alpha^{ij}) \left(\sum_{|\beta| \leq m} D^\beta u D^\beta u_{x^i} \right) \left(\sum_{|\gamma| \leq m} D^\gamma u D^\gamma u_{x^j} \right) \, dx \\
&\quad + 2 \sum_{|\beta| \leq m} \sum_{\beta} \int_{\mathbb{R}^d} G' D^\beta u D^{\beta_1} (a^{ij} - 2\alpha^{ij}) D^{\beta_2} u_{x^i x^j} \, dx.
\end{aligned} \tag{5.5}$$

For the remaining four terms containing the function r , first by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
R_1 &:= 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' D^\beta u D^\beta (\nu^k r^k) \, dx \leq C \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \Psi^{\frac{1}{2}} \|D^\beta r\| \, dx \\
&\leq \frac{\varepsilon}{3} I + \frac{C}{\varepsilon} J.
\end{aligned} \tag{5.6}$$

By the integration by parts, it follows

$$\begin{aligned}
R_2 &:= 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' D^\beta u D^\beta (\sigma^{ik} r_{x^i}^k) dx \\
&\leq 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' D^\beta u (\sigma^{ik} D^\beta r^k)_{x^i} dx + \frac{\varepsilon}{3} I + \frac{C}{\varepsilon} J \\
&= -4 \int_{\mathbb{R}^d} G'' \left[\sum_{|\beta| \leq m} (D^\beta u) \sigma^{ik} D^\beta u_{x^i} \right] \left(\sum_{|\gamma| \leq m} D^\gamma u D^\gamma r^k \right) dx \\
&\quad - 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \sigma^{ik} D^\beta u_{x^i} D^\beta r^k dx + \frac{\varepsilon}{3} I + \frac{C}{\varepsilon} J.
\end{aligned} \tag{5.7}$$

Note that $D^\beta (\sigma^{ik} u_{x^i}) - \sigma^{ik} D^\beta u_{x^i}$ does not contain the $m+1$ order derivatives of u , hence using the integration by parts again we have

$$\begin{aligned}
R_3 &:= - \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \|D^\beta (r - \sigma^i u_{x^i})\|^2 dx \\
&= - \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \|D^\beta r - \sigma^i D^\beta u_{x^i} - [D^\beta (\sigma^i u_{x^i}) - \sigma^i D^\beta u_{x^i}]\|^2 dx \\
&\leq -I - \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \|\sigma^i D^\beta u_{x^i}\|^2 dx + 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \sigma^{ik} D^\beta u_{x^i} D^\beta r^k dx \\
&\quad - 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \sigma^{jk} D^\beta u_{x^j} [D^\beta (\sigma^{ik} u_{x^i}) - \sigma^{ik} D^\beta u_{x^i}] dx + \frac{\varepsilon}{3} I + \frac{C}{\varepsilon} J \\
&\leq -I - \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \|\sigma^i D^\beta u_{x^i}\|^2 dx + 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \sigma^{ik} D^\beta u_{x^i} D^\beta r^k dx \\
&\quad + 4 \int_{\mathbb{R}^d} G'' \left\{ \sum_{|\beta| \leq m} D^\beta u [D^\beta (\sigma^{ik} u_{x^i}) - \sigma^{ik} D^\beta u_{x^i}] \right\} \left(\sum_{|\gamma| \leq m} \sigma^{jk} D^\gamma u D^\gamma u_{x^j} \right) dx \\
&\quad + 2 \sum_{|\beta| \leq m} \sum_{\beta} \int_{\mathbb{R}^d} G' \sigma^{jk} D^\beta u D^{\beta_1} \sigma^{ik} D^{\beta_2} u_{x^i x^j} dx + \frac{\varepsilon}{3} I + \frac{C}{\varepsilon} J.
\end{aligned} \tag{5.8}$$

The last term in (3.4) can be handled as follows:

$$\begin{aligned}
R_4 &:= -2 \int_{\mathbb{R}^d} G'' \left\| \sum_{|\beta| \leq m} D^\beta u D^\beta (r - \sigma^i u_{x^i}) \right\|^2 dx \\
&= -2 \int_{\mathbb{R}^d} G'' \left\| \sum_{|\beta| \leq m} D^\beta u \left(\left\{ D^\beta r - [D^\beta (\sigma^i u_{x^i}) - \sigma^i D^\beta u_{x^i}] \right\} - \sigma^i D^\beta u_{x^i} \right) \right\|^2 dx \\
&\leq -2 \int_{\mathbb{R}^d} G'' \left\| \sum_{|\beta| \leq m} \sigma^i D^\beta u D^\beta u_{x^i} \right\|^2 dx \\
&\quad + 4 \int_{\mathbb{R}^d} G'' \left[\sum_{|\beta| \leq m} (D^\beta u) \sigma^{ik} D^\beta u_{x^i} \right] \left(\sum_{|\gamma| \leq m} D^\gamma u D^\gamma r^k \right) dx \\
&\quad - 4 \int_{\mathbb{R}^d} G'' \left\{ \sum_{|\beta| \leq m} D^\beta u [D^\beta (\sigma^{ik} u_{x^i}) - \sigma^{ik} D^\beta u_{x^i}] \right\} \left(\sum_{|\gamma| \leq m} \sigma^{jk} D^\gamma u D^\gamma u_{x^j} \right) dx.
\end{aligned} \tag{5.9}$$

Then taking (5.6)–(5.9) into account and keeping $\alpha^{ij} = \frac{1}{2} \sigma^{ik} \sigma^{jk}$ in mind, we obtain

$$\begin{aligned}
&R_1 + R_2 + R_3 + R_4 \\
&\leq -(1 - \varepsilon)I + \frac{C}{\varepsilon}J - \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \|\sigma^i D^\beta u_{x^i}\|^2 dx - 2 \int_{\mathbb{R}^d} G'' \left\| \sum_{|\beta| \leq m} \sigma^i D^\beta u D^\beta u_{x^i} \right\|^2 dx \\
&\quad + 2 \sum_{|\beta| \leq m} \sum_{\beta} \int_{\mathbb{R}^d} G' \sigma^{jk} D^\beta u D^{\beta_1} \sigma^{ik} D^{\beta_2} u_{x^i x^j} dx \\
&= -(1 - \varepsilon)I + \frac{C}{\varepsilon}J - 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' \alpha^{ij} D^\beta u_{x^i} D^\beta u_{x^j} dx \\
&\quad - 4 \int_{\mathbb{R}^d} G'' \alpha^{ij} \left(\sum_{|\beta| \leq m} D^\beta u D^\beta u_{x^i} \right) \left(\sum_{|\gamma| \leq m} D^\gamma u D^\gamma u_{x^j} \right) dx \\
&\quad + 2 \sum_{|\beta| \leq m} \sum_{\beta} \int_{\mathbb{R}^d} G' D^\beta u D^{\beta_1} \alpha^{ij} D^{\beta_2} u_{x^i x^j} dx.
\end{aligned} \tag{5.10}$$

By (5.2)–(5.5), (5.10) and the parabolicity condition **(P)** (i.e. the matrix $(a^{ij} - \alpha^{ij})_{d \times d} \geq$

0), we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \Theta(u, r, f, x, t, \omega) dx \\
& \leq -(1 - \varepsilon)I + (C + \frac{C}{\varepsilon})J - 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G'(a^{ij} - \alpha^{ij}) D^\beta u_{x^i} D^\beta u_{x^j} dx \\
& \quad - 4 \int_{\mathbb{R}^d} G''(a^{ij} - \alpha^{ij}) \left(\sum_{|\beta| \leq m} D^\beta u D^\beta u_{x^i} \right) \left(\sum_{|\gamma| \leq m} D^\gamma u D^\gamma u_{x^j} \right) dx \\
& \quad + 2 \sum_{|\beta| \leq m} \sum_{\beta} \int_{\mathbb{R}^d} G' D^\beta u D^{\beta_1} (a^{ij} - \alpha^{ij}) D^{\beta_2} u_{x^i x^j} dx + \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' |D^\beta f|^2 dx \quad (5.11) \\
& \leq -(1 - \varepsilon)I + (C + \frac{C}{\varepsilon})J - 2 \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' A^{ij} D^\beta u_{x^i} D^\beta u_{x^j} dx \\
& \quad + 2 \sum_{|\beta| \leq m} \sum_{\beta} \int_{\mathbb{R}^d} G' D^\beta u D^{\beta_1} A^{ij} D^{\beta_2} u_{x^i x^j} dx + \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' |D^\beta f|^2 dx,
\end{aligned}$$

with $A^{ij} := a^{ij} - \alpha^{ij}$.

To proceed a further estimate to (5.11), we need the following well-known lemma.

Lemma 5.1. (*Oleinik [20]*) Assume that $b^{ij}(x)\xi^i\xi^j \geq 0$ for all $x \in \mathbb{R}^d$, $\xi = (\xi^1, \dots, \xi^d) \in \mathbb{R}^d$ and $b^{ij} \in C^2(\mathbb{R}^d)$. Then for any function $v \in C^2(\mathbb{R}^d)$,

$$(b_{x^\rho}^{ij} v_{x^i x^j})^2 \leq C' b^{ij} v_{x^i x^k} v_{x^j x^k}, \quad \rho = 1, \dots, d,$$

where C' depends only on the second order derivatives of b^{ij} .

By Lemma 5.1, in view of the twice differentiability of A^{ij} and the condition $(A^{ij})_{d \times d} \geq 0$, we have

$$(D^{\beta_1} A^{ij} D^{\beta_2} u_{x^i x^j})^2 \leq C \sum_{|\gamma|=1} A^{ij} D^{\gamma+\beta_2} u_{x^i} D^{\gamma+\beta_2} u_{x^j} \leq C \sum_{|\beta| \leq m} A^{ij} D^\beta u_{x^i} D^\beta u_{x^j}.$$

Thus,

$$2 \int_{\mathbb{R}^d} G' D^\beta u D^{\beta_1} A^{ij} D^{\beta_2} u_{x^i x^j} dx \leq \varepsilon \int_{\mathbb{R}^d} G' (D^{\beta_1} A^{ij} D^{\beta_2} u_{x^i x^j})^2 dx + \frac{C}{\varepsilon} J.$$

Therefore, (5.11) can be simplified as below:

$$\begin{aligned}
& \int_{\mathbb{R}^d} \Theta(u, r, f; x, t, \omega) dx \\
& \leq -(1 - \varepsilon)I - (2 - C\varepsilon) \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' A^{ij} D^\beta u_{x^i} D^\beta u_{x^j} dx \\
& \quad + (C + \frac{C}{\varepsilon})J + \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} G' |D^\beta f|^2 dx.
\end{aligned}$$

Noting the fact that $(A^{ij})_{d \times d} \geq 0$ and ε can be chosen sufficiently small, we obtain the estimate (3.6) and the proof of Lemma 3.1 is complete.

6. Proof of Lemma 3.2

We need do some preparations before proving Lemma 3.2. By condition **(B)** and Sobolev's imbedding theorem, there exists modifications of f and φ , still denoted by f and φ , which are infinitely differentiable w.r.t. x for all (ω, t) . Since the imbedding of $W^{m,2}$ in C^n is continuous when $2(m-n) > d$, the measurability is preserved. Therefore, f are \mathcal{F}_t -measurable for each (t, x) and predictable for each x .

In view of Theorem 2.3 in Du and Meng [6], we know that under conditions **(SP)** and **(B)**, BSPDE (2.1)-(2.2) has a unique generalized solution (u, q) satisfying

$$u \in \bigcap_{m \in \mathbb{Z}_+} L^2_{\mathcal{P}} CW^{m,2}, \quad q \in \bigcap_{m \in \mathbb{Z}_+} L^2_{\mathcal{P}} W^{m,2}. \quad (6.1)$$

We always choose appropriate versions of u and q such that

- (i) for each ω , $u(\omega) \in C([0, T]; W^{m,2})$ for all $m \in \mathbb{Z}_+$;
- (ii) for all t , $\mathbb{E} \|q(t)\|_{m,2}^2 < \infty$ for all $m \in \mathbb{Z}_+$.

By Sobolev's imbedding theorem again, we have the following properties to (u, q) :

- (i) $u(t, x)$ and $q(t, x)$ are \mathcal{F}_t -measurable for each (t, x) ;
- (ii) $u(t, x)$ is jointly continuous in (t, x) for each ω ;
- (iii) $u(t, x)$ and $q(t, x)$ are infinitely differentiable w.r.t. x for each (ω, t) and all the derivatives of $u(t, x)$ are continuous in (t, x) for each ω .

Moreover, we first have the following

Lemma 6.1. *Under conditions **(SP)** and **(B)**, BSPDE (2.1)-(2.2) has a unique generalized solution (u, q) satisfying (6.1) and*

- (a) *for each x , the equation*

$$\begin{aligned} D^\alpha u(t, x) = & D^\alpha \varphi(x) + \int_t^T D^\alpha (a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu \\ & + \sigma^{ik} q_{x^i}^k + \nu^k q^k + f)(s, x) ds - \int_t^T D^\alpha q^k(s, x) dW_s^k \end{aligned} \quad (6.2)$$

holds for all $t \in [0, T]$ and all multi-index α on a full-measure set independent of (t, x) ;

- (b) *for arbitrary $m \in \mathbb{Z}_+$ and $p \geq 2$, $u(\omega) \in C([0, T]; W^{m,p})$ for each ω .*

Proof. Assertion (a) can be derived immediately by (2.8) and the exchange of operators S_ε and D^α . Assertion (b) can be concluded easily by Sobolev's imbedding theorem. For this, recalling that $u(\omega) \in \bigcap_{m \geq 0} C([0, T]; W^{m,2})$ for each ω , by Sobolev's theorem we can find an n such that $W^{n,2}$ is continuously imbedded into $W^{m,p}$. Therefore $u(\omega) \in C([0, T]; W^{m,p})$ for each ω and the lemma is proved. \square

Proof of Lemma 3.2. Step 1. For positive integers M, N , we define

$$H_M(x) = x^2 I_{\{-M \leq x < M\}} + M(2x - M) I_{\{x \geq M\}} - M(2x + M) I_{\{x < -M\}}$$

and

$$G_{N,p}(x) = x^p I_{\{0 \leq x < N\}} + N^{p-1} [px - (p-1)N] I_{\{x \geq N\}}.$$

It is clear that $H_M \in C^{1,1}(\mathbb{R})$, $G_{N,p} \in C^{1,1}[0, \infty)$ satisfying $G(s), G'(s) > 0$ and $G''(s) \geq 0$ for all $s \in (0, \infty)$, and

$$H_M(x) \uparrow x^2, \quad H'_M(x) \rightarrow 2x; \quad G_{N,p}(x) \uparrow x^p, \quad G'_{N,p}(x) \uparrow px^{p-1},$$

as $M, N \rightarrow \infty$, respectively.

Moreover, $G_{N,p}$ satisfies that for all $x \geq 0$,

$$xG'_{N,p}(x) \leq pG_{N,p}(x), \quad |G'_{N,p}(x)|^{\frac{p}{p-1}} \leq p^{\frac{p}{p-1}} G_{N,p}(x). \quad (6.3)$$

For simplicity, we denote

$$\Psi_M := \sum_{|\alpha| \leq m} H_M(D^\alpha u).$$

Doing the generalized Itô's formula (c.f. [9]) to

$$e^{Ks} G_{N,p}(\Psi_M),$$

for each x , by (3.2) and (6.2) we have

$$\begin{aligned} & e^{Kt} G_{N,p}(\Psi_M) + K \int_t^T e^{Ks} G_{N,p}(\Psi_M) ds \\ &= e^{KT} G_{N,p} \left(\sum_{|\alpha| \leq m} H_M(D^\alpha \varphi) \right) + \int_t^T e^{Ks} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta(\mathbb{D}u) ds \\ & \quad - \int_t^T e^{Ks} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} I_{\{-M \leq D^\beta u < M\}} \|D^\beta(r - \sigma^i u_{x^i})\|^2 ds \\ & \quad - \frac{1}{2} \int_t^T e^{Ks} G''_{N,p}(\Psi_M) \left\| \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta(r^k - \sigma^{ik} u_{x^i}) \right\|^2 ds \\ & \quad - \int_t^T e^{Ks} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta(r^k - \sigma^{ik} u_{x^i}) dW_s^k, \end{aligned} \quad (6.4)$$

where we denote

$$\mathbb{D}u := (a^{ij} - 2\alpha^{ij})u_{x^i x^j} + \tilde{b}^i u_{x^i} + cu + \sigma^{ik} r_{x^i}^k + \nu^k r^k + f.$$

Integrating both sides of (6.4) w.r.t. x in \mathbb{R}^d and noting that

$$G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} H'_M(D^\beta u)$$

are bounded, by the stochastic Fubini theorem we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} e^{Kt} G_{N,p}(\Psi_M) dx + K \int_t^T \int_{\mathbb{R}^d} e^{Ks} G_{N,p}(\Psi_M) ds \\
&= \int_{\mathbb{R}^d} e^{KT} G_{N,p} \left(\sum_{|\alpha| \leq m} H_M(D^\alpha \varphi) \right) dx \\
&+ \int_t^T \int_{\mathbb{R}^d} e^{Ks} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta(\mathbb{D}u) dx ds \\
&- \int_t^T \int_{\mathbb{R}^d} e^{Ks} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} I_{\{-M \leq D^\beta u < M\}} \|D^\beta(r - \sigma^i u_{x^i})\|^2 dx ds \\
&- \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G''_{N,p}(\Psi_M) \left\| \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta(r^k - \sigma^{ik} u_{x^i}) \right\|^2 dx ds \\
&- \int_t^T \int_{\mathbb{R}^d} e^{Ks} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta(r^k - \sigma^{ik} u_{x^i}) dx dW_s^k. \tag{6.5}
\end{aligned}$$

First note that $G_{N,p}(\Psi_M) \leq N^{p-1} p \Psi$, hence $|\int_{\mathbb{R}^d} G_{N,p}(\Psi_M) dx|$ is bounded. Similarly, $\int_{\mathbb{R}^d} G_{N,p}(\sum_{|\alpha| \leq m} H_M(D^\alpha \varphi)) dx$ is bounded. On the other hand, both $G'_{N,p}(\Psi_M)$ and $H'_M(D^\beta u)$ are bounded, thus the above stochastic integral is a martingale. Taking first the expectation and then the limit of M for all terms in (6.5), we have

$$\begin{aligned}
& \mathbb{E} \int_{\mathbb{R}^d} e^{Kt} G_{N,p}(\Psi) dx + K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G_{N,p}(\Psi) dx ds \\
&= \mathbb{E} \int_{\mathbb{R}^d} e^{KT} G_{N,p} \left(\sum_{|\alpha| \leq m} |D^\alpha \varphi|^2 \right) dx \\
&+ 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G'_{N,p}(\Psi) \sum_{|\beta| \leq m} D^\beta u D^\beta(\mathbb{D}u) dx ds \\
&- \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G'_{N,p}(\Psi) \sum_{|\beta| \leq m} \|D^\beta(r - \sigma^i u_{x^i})\|^2 dx ds \\
&- \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G''_{N,p}(\Psi) \left\| \sum_{|\beta| \leq m} D^\beta u D^\beta(r^k - \sigma^{ik} u_{x^i}) \right\|^2 dx ds.
\end{aligned}$$

Thus it follows from Lemma 3.1 that

$$\begin{aligned}
& \mathbb{E} \int_{\mathbb{R}^d} e^{Kt} G_{N,p}(\Psi) dx + K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G_{N,p}(\Psi) dx ds \\
&= \mathbb{E} \int_{\mathbb{R}^d} e^{KT} G_{N,p} \left(\sum_{|\alpha| \leq m} |D^\alpha \varphi|^2 \right) dx + \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} \Theta(u, r, f, x, s, \omega) dx ds \\
&\leq \mathbb{E} \int_{\mathbb{R}^d} e^{KT} G_{N,p} \left(\sum_{|\alpha| \leq m} |D^\alpha \varphi|^2 \right) dx - (1 - \varepsilon) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G'_{N,p}(\Psi) \Upsilon dx ds \\
&\quad + \frac{C}{\varepsilon} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G_{N,p}(\Psi) dx ds + \frac{C}{\varepsilon} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G'_{N,p}(\Psi) \Psi dx ds \\
&\quad + C \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G'_{N,p}(\Psi) \sum_{|\beta| \leq m} |D^\beta f|^2 dx ds,
\end{aligned}$$

where the function Θ is defined by (3.4) w.r.t. $G_{N,p}$ instead of G . Then, in view of (6.3) and by Young's inequality, we have

$$\begin{aligned}
& \mathbb{E} \int_{\mathbb{R}^d} e^{Kt} G_{N,p}(\Psi) dx + \left(K - \frac{C}{\varepsilon} - \frac{Cp}{\varepsilon} - C \right) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G_{N,p}(\Psi) dx ds \\
&+ (1 - \varepsilon) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Ks} G'_{N,p}(\Psi) \Upsilon dx ds \\
&\leq \mathbb{E} \int_{\mathbb{R}^d} e^{KT} G_{N,p} \left(\sum_{|\alpha| \leq m} |D^\alpha \varphi|^2 \right) dx + C e^{KT} \left(\frac{p-1}{p} \right)^{p-1} \mathbb{E} \int_t^T \|f\|_{m,2p}^{2p} ds. \quad (6.6)
\end{aligned}$$

Choosing

$$\varepsilon = \frac{1}{2} \quad \text{and} \quad K = 3C + 2Cp + 1,$$

letting $N \rightarrow \infty$ in (6.6) and using the monotone convergence, we have

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \Psi^p dx ds + \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \Psi^{p-1} \Upsilon dx ds \\
&\leq C e^{Cp} \mathbb{E} \|\varphi\|_{m,2p}^{2p} + C e^{Cp} \mathbb{E} \int_0^T \|f\|_{m,2p}^{2p} ds < \infty. \quad (6.7)
\end{aligned}$$

Step 2. We estimate the term $\mathbb{E} \sup_{t \leq T} \int_{\mathbb{R}^d} \Psi^p dx$. By the BDG inequality, it turns out

that

$$\begin{aligned}
& \mathbb{E} \sup_{t \leq T} \left| \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta (r^k - \sigma^{ik} u_{x^i}) dx dW_s^k \right| \\
& \leq C \mathbb{E} \left[\left\{ \int_0^T \left\| \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta (r - \sigma^i u_{x^i}) dx \right\|^2 ds \right\}^{\frac{1}{2}} \right] \\
& \leq C \mathbb{E} \left[\left\{ \int_0^T \left(\int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} |H'_M(D^\beta u)| \sum_{|\gamma| \leq m} \|D^\gamma r\| dx \right)^2 \right. \right. \\
& \quad \left. \left. + \left(\int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} |H'_M(D^\beta u)| \sum_{|\gamma| \leq m} |D^\gamma u| dx \right)^2 \right. \right. \\
& \quad \left. \left. + \left(\int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} H'_M(D^\beta u) \sigma^{ik} D^\beta u_{x^i} dx \right)^2 ds \right\}^{\frac{1}{2}} \right] \\
& \leq C \mathbb{E} \left[\left\{ \int_0^T \left(\int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi_M dx \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Upsilon dx \right. \right. \right. \\
& \quad \left. \left. + \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi_M dx \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi dx \right. \right. \\
& \quad \left. \left. + \left| \int_{\mathbb{R}^d} \{G_{N,p}(\Psi_M)\}_{x^i} \sigma^{ik} dx \right|^2 \right) ds \right\}^{\frac{1}{2}} \right] \\
& \leq C \mathbb{E} \sqrt{\left(\sup_{t \leq T} \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi_M dx \right) \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Upsilon dx ds} \\
& \quad + C \mathbb{E} \sqrt{\left(\sup_{t \leq T} \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi_M dx \right) \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi dx ds} \\
& \quad + C \mathbb{E} \sqrt{\left(\sup_{t \leq T} \int_{\mathbb{R}^d} G_{N,p}(\Psi_M) dx \right) \int_0^T \int_{\mathbb{R}^d} G_{N,p}(\Psi_M) dx ds} \\
& \leq \frac{\varepsilon}{p} \mathbb{E} \sup_{t \leq T} \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi_M dx + Cp \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Upsilon dx ds \\
& \quad + \frac{\varepsilon}{p} \mathbb{E} \sup_{t \leq T} \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi_M dx + Cp \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi dx ds \\
& \quad + \varepsilon \mathbb{E} \sup_{t \leq T} \int_{\mathbb{R}^d} G_{N,p}(\Psi_M) dx + C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G_{N,p}(\Psi_M) dx ds.
\end{aligned}$$

Hence, Considering (6.5) with $K = 0$, by BDG inequality and (6.3) we have

$$\begin{aligned}
& (1 - 3\varepsilon)\mathbb{E} \sup_{t \leq T} \int_{\mathbb{R}^d} G_{N,p}(\Psi_M) dx \\
\leq & \mathbb{E} \int_{\mathbb{R}^d} G_{N,p} \left(\sum_{|\alpha| \leq m} H_M(D^\alpha \varphi) \right) dx \\
& + \mathbb{E} \int_0^T \left| \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta (\mathbb{D}u) dx \right. \\
& \quad - \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \sum_{|\beta| \leq m} I_{\{-M \leq D^\beta u < M\}} \|D^\beta(r - \sigma^i u_{x^i})\|^2 dx \\
& \quad \left. - \frac{1}{2} \int_{\mathbb{R}^d} G''_{N,p}(\Psi_M) \left\| \sum_{|\beta| \leq m} H'_M(D^\beta u) D^\beta(r^k - \sigma^{ik} u_{x^i}) \right\|^2 dx \right| ds \\
& + Cp \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Upsilon dx ds + Cp \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi_M) \Psi dx ds \\
& + C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G_{N,p}(\Psi_M) dx ds.
\end{aligned}$$

Taking the limit of M , by Lemma 3.1 and (6.3) again we have

$$\begin{aligned}
& (1 - 3\varepsilon)\mathbb{E} \sup_{t \leq T} \int_{\mathbb{R}^d} G_{N,p}(\Psi) dx \\
\leq & \mathbb{E} \int_{\mathbb{R}^d} G_{N,p} \left(\sum_{|\alpha| \leq m} |D^\alpha \varphi|^2 \right) dx + \mathbb{E} \int_0^T \left| \int_{\mathbb{R}^d} \Theta(u, r, f, x, s, \omega) dx \right| ds \\
& + Cp \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi) \Upsilon dx ds + Cp \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi) \Psi dx ds \\
& + C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G_{N,p}(\Psi) dx ds \\
\leq & \mathbb{E} \int_{\mathbb{R}^d} G_{N,p} \left(\sum_{|\alpha| \leq m} |D^\alpha \varphi|^2 \right) dx + \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \sum_{|\beta| \leq m} G'(\Psi) |D^\beta f|^2 dx ds \\
& + (Cp + 1) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi) \Upsilon dx ds + \left(\frac{C}{\varepsilon} + Cp\right) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi) \Psi dx ds \\
& + \left(\frac{C}{\varepsilon} + C\right) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G_{N,p}(\Psi) dx ds \\
\leq & Ce^{Cp} \mathbb{E} \|\varphi\|_{m,2p}^{2p} + Ce^{Cp} \mathbb{E} \int_t^T \|f\|_{m,2p}^{2p} ds + (Cp + 1) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G'_{N,p}(\Psi) \Upsilon dx ds \\
& + \left(\frac{Cp}{\varepsilon} + Cp^2 + \frac{C}{\varepsilon} + C\right) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} G_{N,p}(\Psi) dx ds. \tag{6.8}
\end{aligned}$$

Then choosing $\varepsilon = \frac{1}{6}$ and letting $N \rightarrow \infty$ in (6.8), in view of (6.7), we have

$$\begin{aligned} \mathbb{E} \sup_{t \leq T} \int_{\mathbb{R}^d} \Psi^p dx &\leq C e^{Cp} \mathbb{E} \|\varphi\|_{m,2p}^{2p} + C e^{Cp} \mathbb{E} \int_0^T \|f\|_{m,2p}^{2p} ds \\ &\quad + C p^2 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \Psi^p dx ds + C p \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \Psi^{p-1} \Upsilon dx ds \\ &\leq C e^{Cp} \mathbb{E} \|\varphi\|_{m,2p}^{2p} + C e^{Cp} \mathbb{E} \int_0^T \|f\|_{m,2p}^{2p} ds < \infty. \end{aligned}$$

The proof of Lemma 3.2 is complete. \square

References

- [1] R.A. Adams, Sobolev Spaces, Academic Press, 1975.
- [2] A. Al-Hussein, Backward stochastic partial differential equations driven by infinite dimensional martingale and applications, Stoch. Stoch. Rep. 81 (2009) 601–626.
- [3] V. Barbu, A. Răşcanu, G. Tessitore, Carleman estimates and controllability of linear stochastic heat equations, Appl. Math. Optim. 47 (2003) 97–120.
- [4] A. Bensoussan, Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions, Stochastics 9 (1983) 169–222.
- [5] M. Chang, Y. Pang, J. Yong, Optimal stopping problem for stochastic differential equations with random coefficients, SIAM J. Control Optim. 48 (2009) 94–971.
- [6] K. Du, Q. Meng, A revisit to W_2^n -theory of super-parabolic backward stochastic partial differential equations in R^d , Stoch. Proc. Appl. 120 (2010) 1996–2015.
- [7] K. Du, J. Qiu, S. Tang, L^p theory for super-parabolic backward stochastic partial differential equations in the whole space, arXiv:1006.1171v1 [math.PR] (2010).
- [8] N. El Karoui, S. Peng, M.C. Quenez, Backward stochastic differential equations in finance, Mathematical Finance 7 (1997) 1–71.
- [9] K.D. Elworthy, A. Truman, H.Z. Zhao, Generalized Itô's formulae and space-time Lebesgue-Stieltjes integrals of local times, Sèminaire de Probabilités 40 (2007) 117–136.
- [10] N. Englezos, I. Karatzas, Utility maximization with habit formation: dynamic programming and stochastic PDEs, SIAM J. Control Optim. 48 (2009) 481–520.
- [11] U.G. Haussmann, The maximum principle for optimal control of diffusions with partial information, SIAM J. Control Optim. 25 (1987) 341–361.
- [12] Y. Hu, J. Ma, J. Yong, On semi-linear degenerate backward stochastic partial differential equations, Probab. Theory Relat. Fields 123 (2002) 381–411.

- [13] N.V. Krylov, B.L. Rozovskii, On the Cauchy problem for linear stochastic partial differential equations, *Math. USSR Izvestija* 11 (1977) 1267–1284.
- [14] N.V. Krylov, B.L. Rozovskii, Stochastic evolution equations, *J. Sov. Math.* 16 (1981) 1233–1277.
- [15] N.V. Krylov, B.L. Rozovskii, Characteristics of degenerating second-order parabolic Itô equations, *Journal of Mathematical Sciences* 32 (1982) 336–348.
- [16] J. Ma, J. Yong, Adapted solution of a degenerate backward SPDE, with applications, *Stoch. Proc. Appl.* 70 (1997) 59–84.
- [17] J. Ma, J. Yong, On linear, degenerate backward stochastic partial differential equations, *Probab. Theory Relat. Fields* 113 (1999) 135–170.
- [18] H. Nagai, Risk-sensitive portfolio optimization with full and partial information, *Advanced Studies in Pure Mathematics* 41 (2004) 257–278.
- [19] N. Nagasa, M. Nisio, Optimal controls for stochastic partial differential equations, *SIAM J. Control Optim.* 28 (1990) 186–213.
- [20] O.A. Oleinik, Linear equations of second order with nonnegative characteristic form, *Amer. Math. Soc. Transl.* 65 (1967) 167–199.
- [21] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, *Systems Control Lett.* 14 (1990) 55–61.
- [22] S. Peng, Stochastic Hamilton-Jacobi-Bellman equations, *SIAM J. Control Optim.* 30 (1992) 284–304.
- [23] B.L. Rozovskii, *Stochastic Evolution Systems, Mathematics and Its Applications*, Kluwer Academic Publishers, 1990.
- [24] S. Tang, The maximum principle for partially observed optimal control of stochastic differential equations, *SIAM J. Control Optim.* 36 (1998) 1596–1617.
- [25] S. Tang, A new partially observed stochastic maximum principle, in: *Proceedings of 37th IEEE Control and Decision Conference*, Tampa, Florida, pp. 2353–2358.
- [26] S. Tang, Investigation into semi-linear systems of backward stochastic partial differential equations with forward and backward stochastic differential equations, preprint (2003) , 51 pages.
- [27] S. Tang, Semi-linear systems of backward stochastic partial differential equations in R^n , *Chinese Annals of Mathematics* 26B (2005) 437–456.
- [28] S. Tang, X. Zhang, Null controllability for forward and backward stochastic parabolic equations, *SIAM Journal on Control and Optimization* 48 (2009) 2191–2216.
- [29] G. Tessitore, Existence, uniqueness and space regularity of the adapted solutions of a backward SPDE, *Stochastic Analysis and Applications* 14 (1996) 461–486.

- [30] Q. Zhang, H.Z. Zhao, Stationary solutions of SPDEs and infinite horizon BDSDEs with non-Lipschitz coefficients, *J. Differ. Equations* 248 (2010) 953–991.
- [31] X. Zhou, A duality analysis on stochastic partial differential equations, *Journal of Functional Analysis* 103 (1992) 275–293.
- [32] X. Zhou, On the necessary conditions of optimal controls for stochastic partial differential equations, *SIAM J. Control Optim.* 31 (1993) 1462–1478.